Neural Proofs for Sound Verification and Control of Complex Systems

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[references at end of deck]

Outline

1. Context
2. Neural Proofs - Formal Synthesis of Neural Certificates
3. Synthesis of Formal Abstractions

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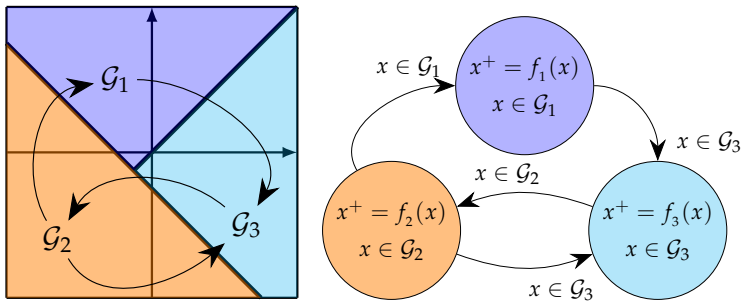
1. Context
2. Neural Proofs - Formal Synthesis of Neural Certificates
3. Synthesis of Formal Abstractions

Control vs formal verification

- dynamical models for control

$$x \in \mathbb{R}^n, \mathcal{G}_i \subset \mathbb{R}^n, i \in \{1,2,3\}$$
$$\forall x \in \mathcal{G}_i, \quad x^+ = f_i(x)$$



- safety,
  reachability,
  stability
- invariants & barrier certificates,
  reach-set computation,
  Lyapunov functions

- software programs

```
int x, c, r;
...
x = rand(1,...,10);
c = 1;
while (x > 0) {
r = Bernoulli(0.5);
if (c == 1 && r) { x = 2 * x; }
else if (c == 1 && !r) { c++; }
else { x = x - 1; }
}
```

- invariance,
  termination,
  assertion check/violation
- program and loop invariants,
  reach over CFG, abstract int,
  ranking functions and progress measures

Models

- dynamical/transition systems and reactive programs

$$x \in \mathbb{S}, \theta \in \Theta, a \in \mathbb{A}$$
$$x^+ = f(x, \theta, a)$$

- x - states/variables (possibly hybrid)
- θ - uncertainty (lack of determinism)
- a/u - action, input, decisions
  - non-environmental: adversarial (demonic) vs agential (angelic)
- general-space MDPs[1] [2] [3], or SHS[4]

[1] Bertsekas & Shreve, "Stochastic optimal control: The discrete-time case," Athena Scientific, 1996

[2] Hernández-Lerma & Lasserre, "Discrete-time Markov control processes," Springer, 1996

[3] Meyn & Tweedie, "Markov chains and stochastic stability," Springer, 1993

[4] A. Lavaei et.al., "Automated Verification and Synthesis of Stochastic Hybrid Systems: A Survey," Automatica, v 146, 2022

Cyber-Physical Systems: from code to control

- complex embedded systems
- interleaving of cyber/digital components with physical/analogue dynamics

- dynamics, computation, and control

- safety-critical applications

$\rightarrow$ sound and automated verification

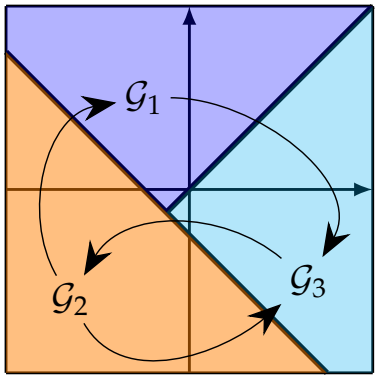
$\rightarrow$ correct-by-design control synthesis

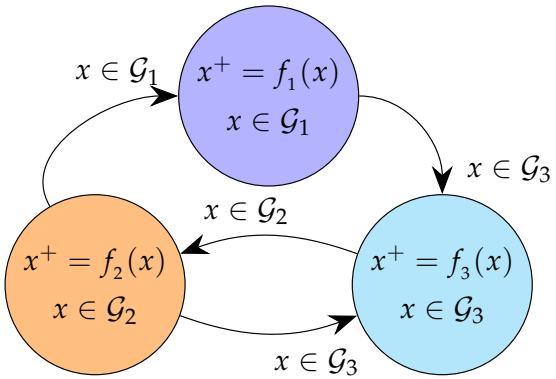
Properties: Encoding rich dynamical behaviour

- as specifications, requirements for verification (e.g., safety)
- as tasks, objectives for control design (e.g., reachability)
- without manual reward engineering - *"rewards are NOT enough"*
  useful also in model-free (e.g., RL) sequential decision making

Properties: Encoding rich dynamical behaviour

$$x \in \mathbb{R}^n$$
$$\mathcal{G}_i \subset \mathbb{R}^n, i \in \{1, \dots m\}$$
$$\forall x \in \mathcal{G}_i, \quad x^+ = f_i(x)$$

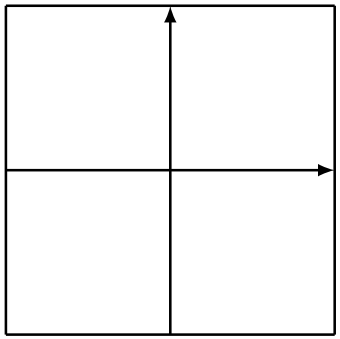




Properties: Encoding rich dynamical behaviour

$$x \in \mathbb{R}^n$$

$$x^+ = f(x)$$



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Properties: Encoding rich dynamical behaviour

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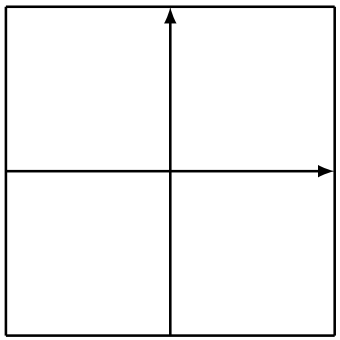
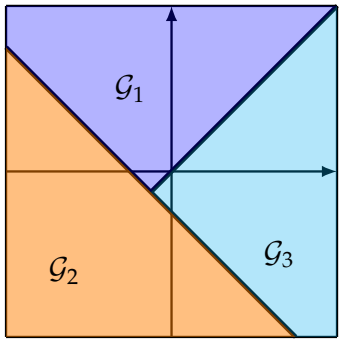




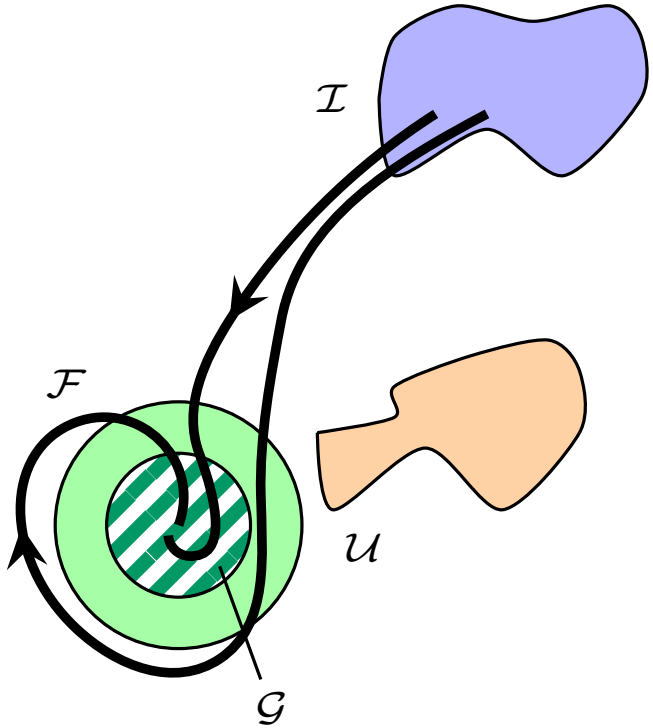
$$\begin{array}{lccccc} x: & x_0 \rightarrow & x_1 \rightarrow & x_2 \rightarrow & \ldots \\ \rho: & \mathcal{G}_{x_0} \rightarrow & \mathcal{G}_{x_1} \rightarrow & \mathcal{G}_{x_2} \rightarrow & \ldots \end{array}$$

- model's behaviour (*cf.* Jan Willems) $\leftrightarrow$ formal language

Properties: Encoding rich dynamical behaviour

- consider (class of) properties/requirements/specifications

$$\forall x_0 \in \mathcal{I}, \quad \exists T \in \mathbb{N}, \quad \forall k \in \{0, 1, \ldots, T-1\}, \quad \forall \tau \geq T : \\ x_T \in \mathcal{G}, \quad x_k \notin \mathcal{U}, \quad x_\tau \in \mathcal{F}$$



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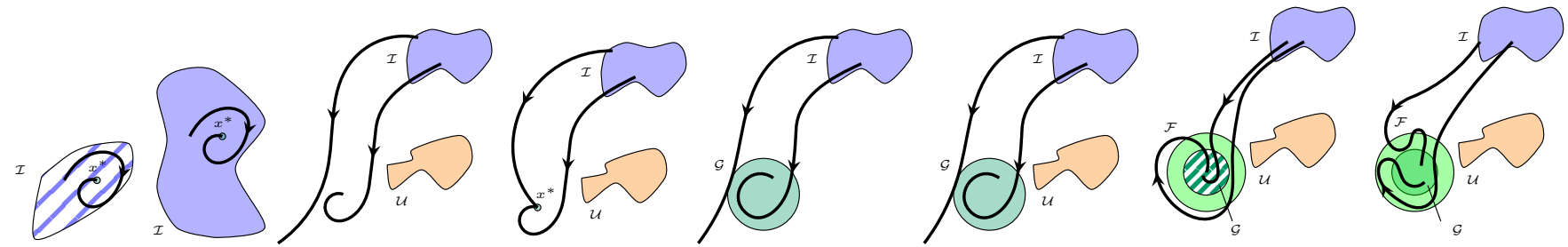

a: Stability b: ROA c: Safety d: SWA e: Reachability f: RWA g: RSWA h: RAR

- class encompasses stability, invariance, safety, reachability, reach-avoid, ... (cf. safe/co-safe fragment in linear temporal logic)

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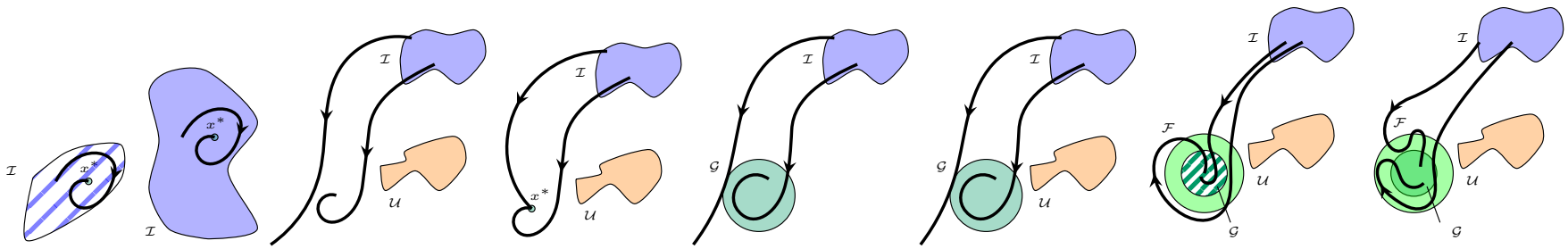
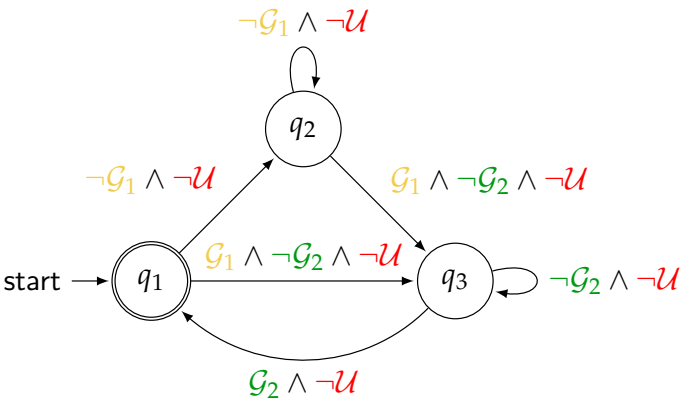


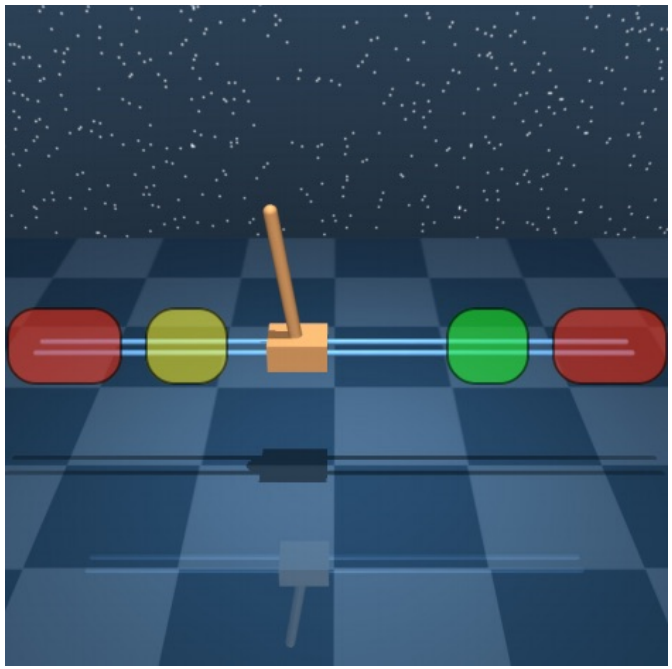
a: Stability b: ROA c: Safety d: SWA e: Reachability f: RWA g: RSWA h: RAR

- connections to:
  - temporal logics
  - formal languages
  - automata theory
- beyond obligation, to full reactivity (∞-horizon specs)

Properties: Encoding rich dynamical behaviour

$$\Box\Diamond\mathcal{G}_1 \wedge \Box\Diamond\mathcal{G}_2 \wedge \Box\neg\mathcal{U}$$





Synthesis for verification and design

from verification/analysis

to model design, or synthesis of controller (scheduler, policy, strategy)

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this line of research:
consider broad model class (∞-state g-MDP's)
propose two general, automated, and formal approaches:

Synthesis for verification and design

from verification/analysis

to model design, or synthesis of controller (scheduler, policy, strategy)

this line of research:
consider broad model class (∞-state g-MDP's)
propose two general, automated, and formal approaches:

1. synthesis of certificates for verification, and of controllers+certificates
2. synthesis of abstractions, for verification and design [ThB03, FrB02]

Synthesis for verification and design

from verification/analysis

to model design, or synthesis of controller (scheduler, policy, strategy)

this line of research:
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Neural Proofs - Formal Synthesis of Neural Certificates

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Decision problems: SAT and SMT

- SAT is a decision problem (yes/no question)
- problem: to find satisfying assignment of Boolean function
- e.g., assume Boolean x_i, check

$$\exists x_1, x_2, x_3 : \quad (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge \neg x_1$$

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- SMT is a decision problem for a logical formula within a theory
- instance: theory of non-linear arithmetics over real closed fields
- e.g., assume reals $x_i \in \mathbb{R}$, check

$$\exists x_1, x_2 : \quad x_1 \geq 0 \Rightarrow 3x_1 + 2x_2 + 1 > 0$$

Decision problems: SAT and SMT

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- SMT is a decision problem for a logical formula within a theory
- consider harder problem (2nd-order logic): assume reals $x_i \in \mathbb{R}$, seek function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, s.t.

$$\exists F(x_1, x_2), \forall x_1, x_2 :$$
$$F(x_1, x_2) \geq x_1 \wedge F(x_1, x_2) \geq x_2 \wedge (F(x_1, x_2) = x_1 \vee F(x_1, x_2) = x_2)$$

(spoiler: $F(x_1, x_2) = \max\{x_1, x_2\}$)

Neural proofs and certificates

neural proofs = 1) proof rules + 2) certificates synthesis

1) proof rules = spec-dependent algebraic conditions on model's dynamics
2) certificate synthesis

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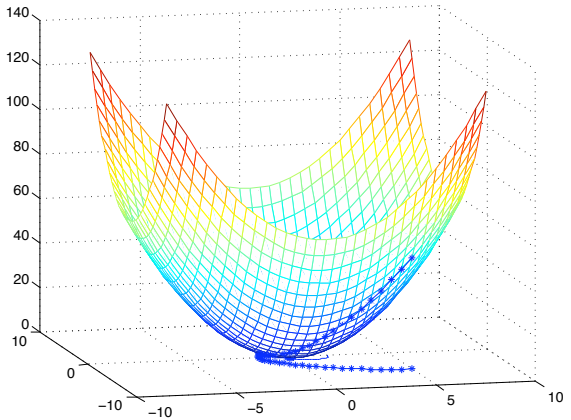
neural proofs = 1) proof rules + 2) certificates synthesis

1) proof rules = spec-dependent algebraic conditions on model's dynamics
2) certificate synthesis, 2a) deductive vs 2b) inductive
   - 2a) deductive, i.e. via logic, SMT
   - 2b) inductive: neural networks + SMT

Lyapunov functions for stability

- consider $x^+ = f(x)$, assume $x_e \in \mathbb{R}^n$ is an equilibrium, $f(x_e) = x_e$
- ensure (asymptotic) stability of $x_e \in \mathcal{D} \subseteq \mathbb{R}^n$ (assume $\mathcal{D}$ given)
- by finding $\mathbb{R}$-valued Lyapunov function $V(x)$, satisfying

1. lower bound:
$$V(x_e) = 0 \tag{1}$$
2. positive definiteness:
$$V(x) > 0,\ \forall x \in \mathcal{D} \setminus \{x_e\} \tag{2}$$
3. negative decrease:
$$V(f(x)) - V(x) < -\epsilon,\ \forall x \in \mathcal{D} \setminus \{x_e\} \quad (\epsilon > 0) \tag{3}$$



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- that is, solve following synthesis problem:

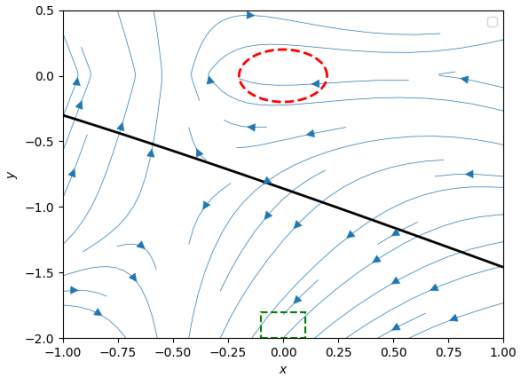
$$\exists V : \mathcal{D} \to \mathbb{R} \quad s.t. \ \ \forall x \in \mathcal{D}, \quad \text{conditions (1)} \wedge \text{(2)} \wedge \text{(3) hold}$$

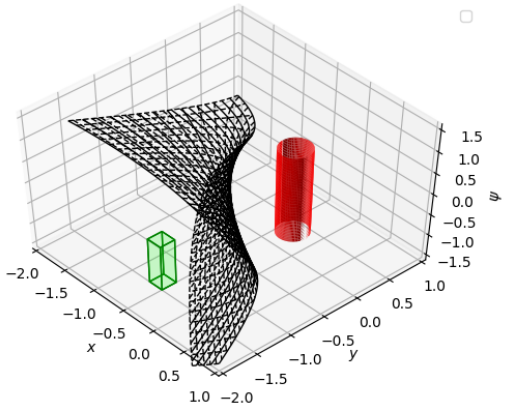
- feed 2nd-order logical formula above to SMT solver

Barrier certificates for safety

- consider $x^+ = f(x)$, along with sets $\mathcal{I}$ (initial) and $\mathcal{U}$ (unsafe)
- ensure there exists no trajectory starting in $\mathcal{I}$ ever entering $\mathcal{U}$

Barrier certificates for safety





Barrier certificates for safety

- consider $x^+ = f(x)$, along with sets $\mathcal{I}$ (initial) and $\mathcal{U}$ (unsafe)
- ensure there exists no trajectory starting in $\mathcal{I}$ ever entering $\mathcal{U}$
- by finding barrier function $B(x)$, satisfying

1. negativity within initial set $\mathcal{I}$:

$$B(x) \leq 0, \ \forall x \in \mathcal{I}$$

2. positivity within unsafe set $\mathcal{U}$:

$$B(x) > 0, \ \forall x \in \mathcal{U}$$

3. set invariance property via negative decrease:

$$B(f(x)) - B(x) < 0, \ \forall x \in (\mathcal{U} \cup \mathcal{I})^C$$

- again, feed a 2nd-order logical formula to SMT solver

Ranking functions for termination/halting/reachability

- consider probabilistic program $\mathtt{x}^+ = f(\mathtt{x}, \mathtt{w})$:

```
while (x > 0):
if prob(0.75) {
x-- }
```

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- ensure that, for any $\mathtt{x} \geq 0$, $\mathsf{F}(\mathtt{x} \leq 0)$ holds almost surely (with probability 1)
- by finding ranking function $V(\mathtt{x})$, satisfying

1. $V(0) = 0$
2. $\mathtt{x} > 0 \rightarrow V(\mathtt{x}) > 0$
3. $\mathtt{x} > 0 \rightarrow \mathbb{E}[V(f(\mathtt{x}))] \leq V(\mathtt{x}) - \epsilon$

   namely, if $\mathtt{x} > 0$, $V(\mathtt{x})$ decreases in expectation by ϵ (positive constant)

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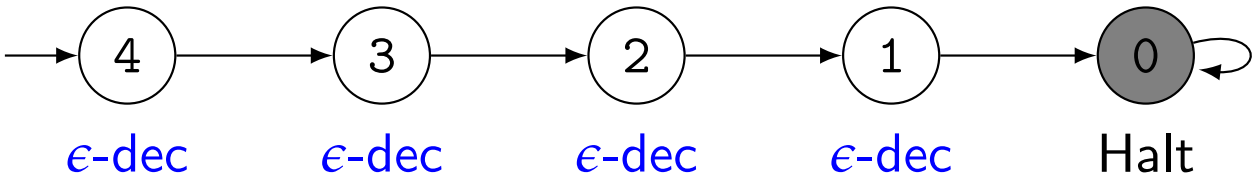
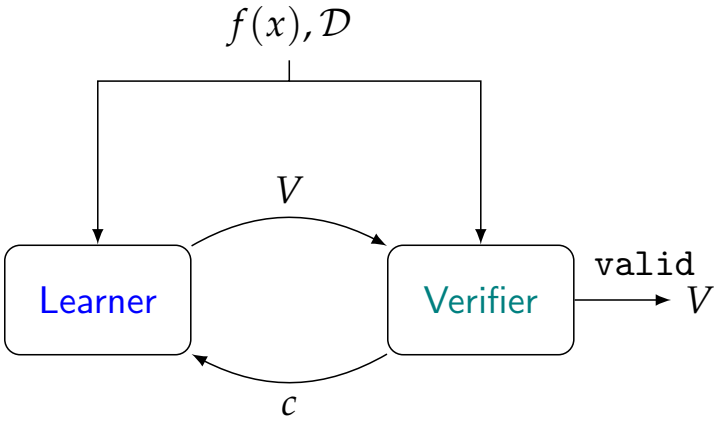


e.g., $V(\mathtt{x}) = \mathtt{x} \geq 0$ & decreases in expectation by say $\epsilon = 3/4$

Counterexample-guided Inductive Synthesis (CEGIS)



1. Learner

   generates candidates V over finite set S

2. Verifier

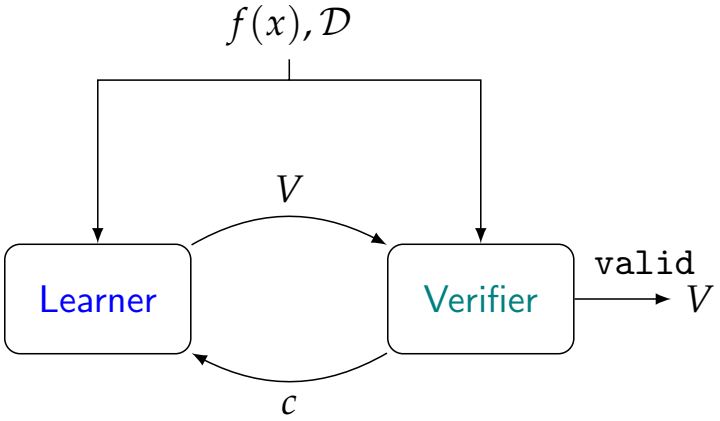
   certifies validity on $\mathcal{D}$, or provides counterexample(s) c

- inductive synthesis loop

  1. sample (finite) set $S \subset \mathcal{D}$
  2. Learner generates $V(\theta, \cdot)$ via query SMT solver on formula: $\exists \theta \in \Theta$: proof rule holds on points $s \in S$
  3. Verifier checks either $V(\cdot, x)$ valid over dense $\mathcal{D}$, or counterexample c : query SMT solver on formula $\exists c \in \mathcal{D}$: proof rule does not hold on c
  4. $S \leftarrow S \cup c$, loop back to 2

- sound, but not complete: infinite search space ($\theta \in \Theta$) and domain ($x \in \mathcal{D}$)

Neural Inductive Synthesis



1. Learner

   generates candidates V over finite set S
2. Verifier

   certifies validity on $\mathcal{D}$, or provides counterexample(s) c

- inductive synthesis loop

  1. sample (finite) set $S \subset \mathcal{D}$
  2. Learner trains neural $V(\theta, \cdot)$ via loss function L;
     L penalises deviations from conditions in proof rules
  3. Verifier checks either $V(\cdot, x)$ valid over dense $\mathcal{D}$, or counterexample c :
     query SMT solver on formula $\exists c \in \mathcal{D}$: proof rule does not hold on c
  4. $S \leftarrow S \cup c$, loop back to 2

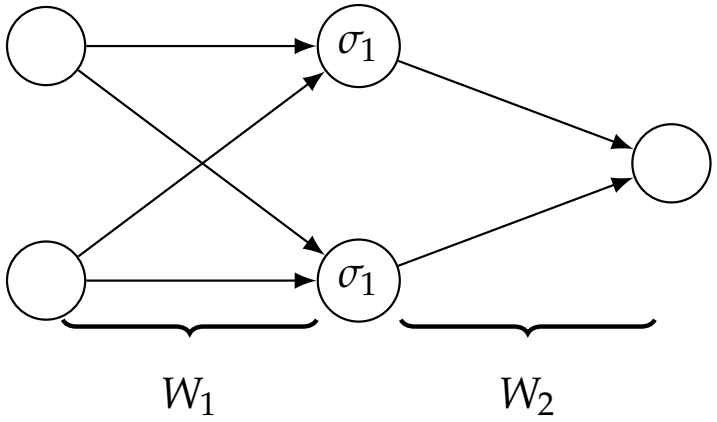
- sound, but not complete: infinite search space ($\theta \in \Theta$) and domain ($x \in \mathcal{D}$)

Neural networks as Lyapunov functions

- neural nets are general and flexible (universal function approximators)
- Learner trains shallow neural network

$$V(x) = W_2 \cdot \sigma_1(W_1 x + b_1)$$

(W_i weights, (σ_1) activation fcns)

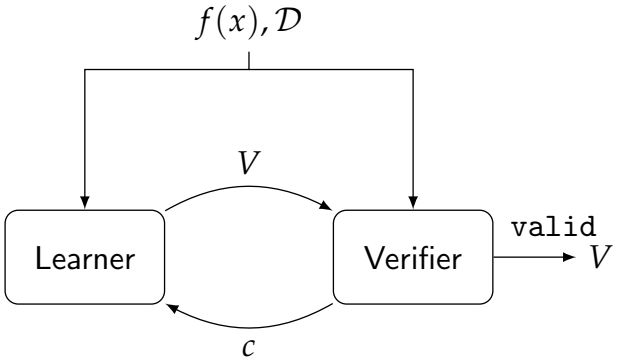


- loss function enforces Lyapunov conditions in (1)+(2) and (3) on points in S:

$$L(S) = \mid V(0) \mid + \sum_{s \in S} \max\{0, -V(s)\} + \sum_{s \in S} \max\{0, V(f(s)) - V(s) + \epsilon\}$$

- loss function L is *"pretty good"* proxy of synthesis formula

Neural networks as Lyapunov functions



- surprisingly effective! Communication *Learner* $\leftrightarrow$ *Verifier* is crucial

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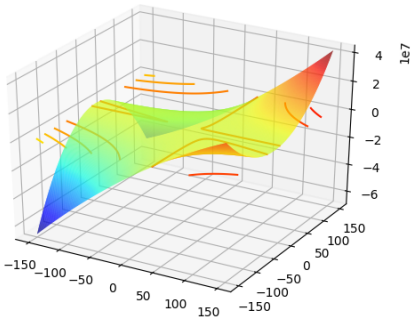
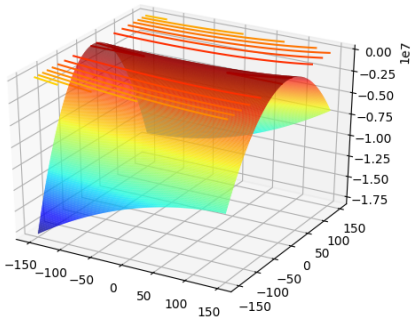
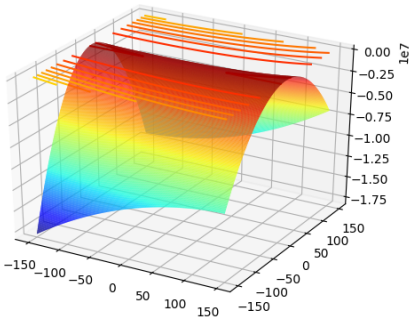
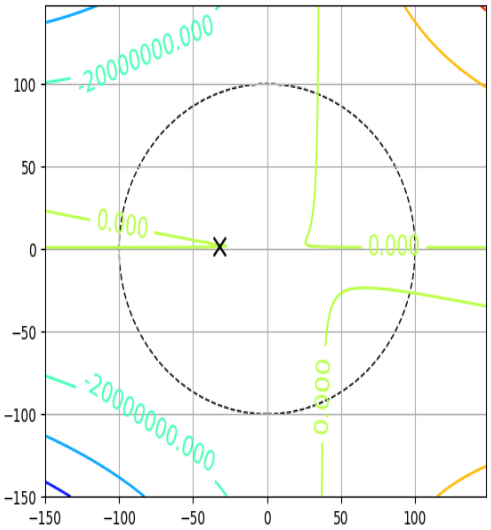
Synthesis of Lyapunov functions - example

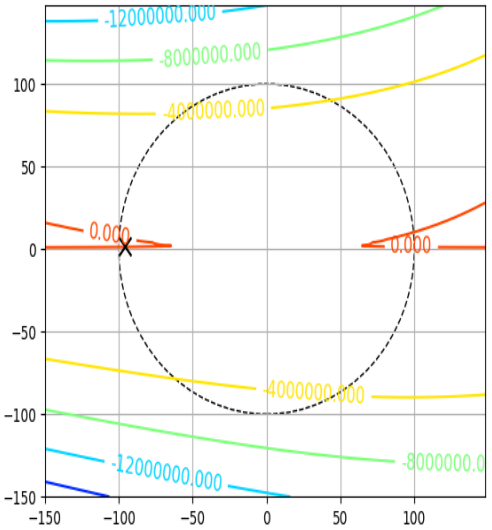








x c-ex



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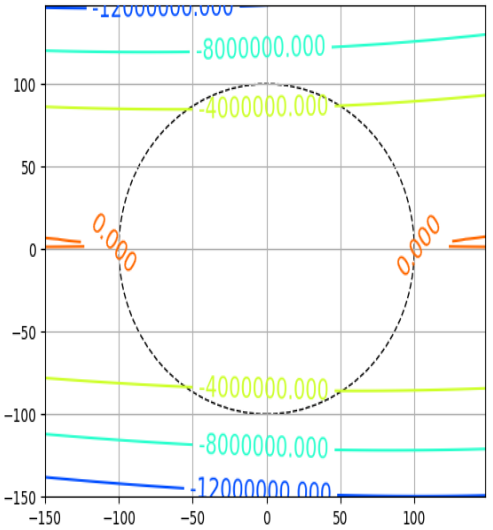
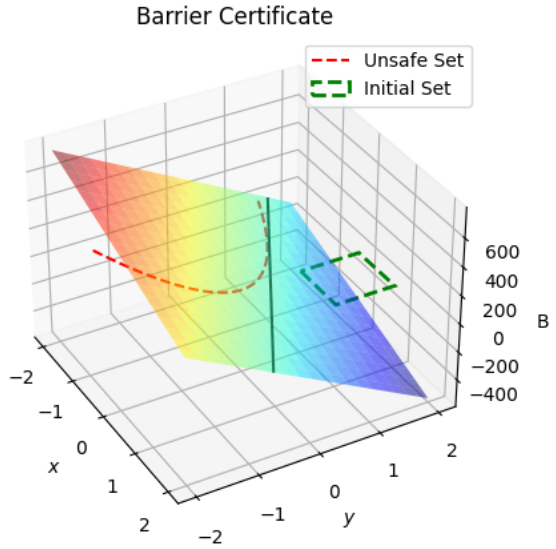
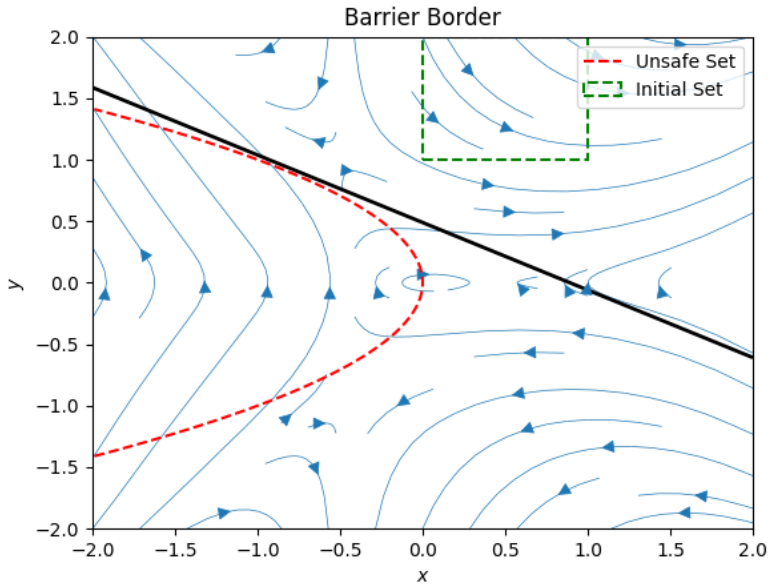


NO c-ex

Synthesis of barrier certificates - examples

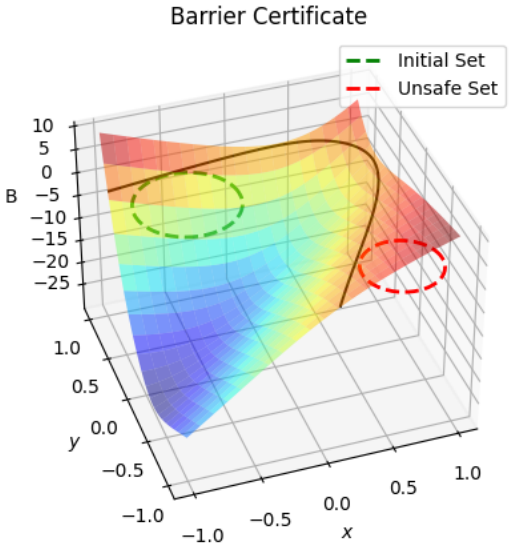


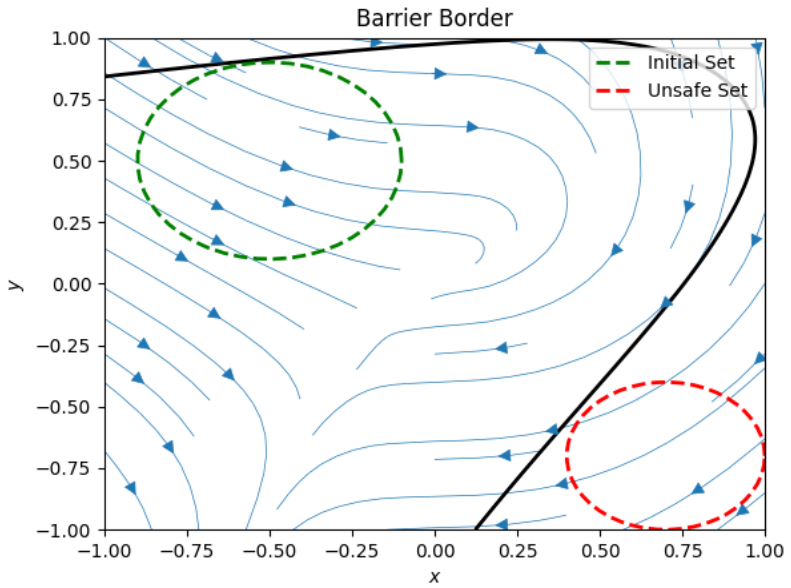


$$\begin{cases} x^+ &= y + 2xy \\ y^+ &= -x + 2x^2 - y^2 \end{cases}$$

[10] · Linear

Synthesis of barrier certificates - examples

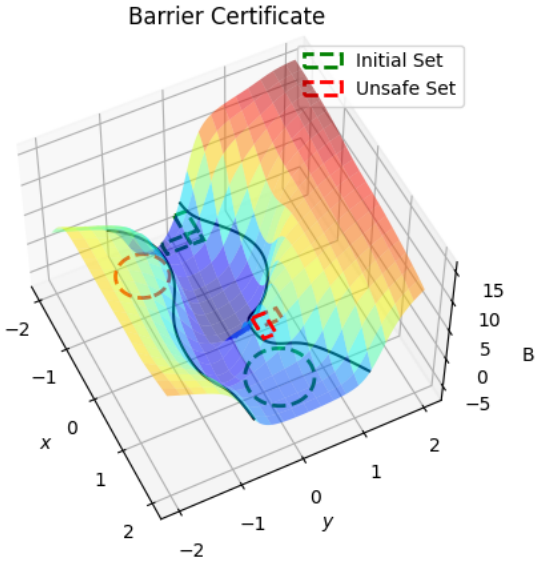


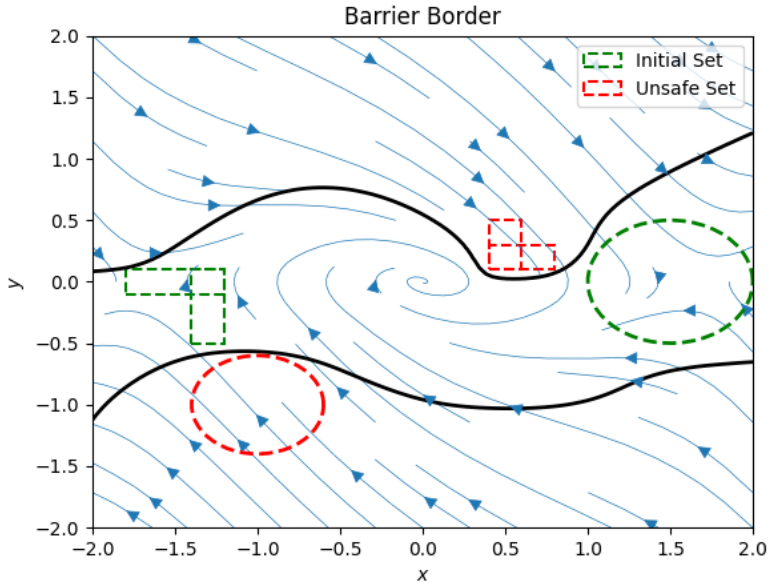


$$\begin{cases} x^+ &= \exp(-x) + y - 1 \\ y^+ &= -\sin(x)^2 \end{cases}$$

[20] · Softplus

Synthesis of barrier certificates - examples





$$\begin{cases} x^+ &= y \\ y^+ &= -x - y + \frac{1}{3}x^3 \end{cases}$$

$[20, 20] \cdot \text{Sigmoid}, \text{Sigmoid}$

Synthesis of control certificates for complex tasks

- dynamical models with inputs (a.k.a., external non-determinism)

$$x^+ = f(x, u)$$

$\rightarrow$ synthesis of "control certificates"

- modify known synthesis problem:

$$\exists V : \mathcal{D} \to \mathbb{R} \quad s.t. \quad \forall x \in \mathcal{D} \quad \text{conditions (1)} \wedge \text{(2)} \wedge \text{(3) hold}$$

Synthesis of control certificates for complex tasks

- dynamical models with inputs (a.k.a., external non-determinism)

$$x^+ = f(x, u)$$

$\rightarrow$ synthesis of "control certificates"

- approach:
  1. control policies are also NN-templated
  2. concurrent synthesis controls and corresponding certificates

Synthesis of control certificates for complex tasks

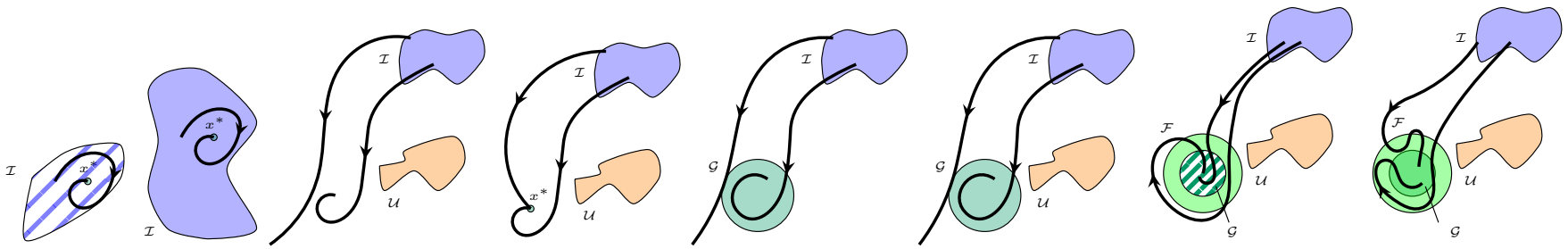
- dynamical models with inputs (a.k.a., external non-determinism)

$$x^+ = f(x, u)$$

$\rightarrow$ synthesis of "control certificates"

- (back to) broad class of properties/requirements

$$\forall x_0 \in \mathcal{I}, \quad \exists T \in \mathbb{N}, \quad \forall t \in \{0, \ldots, T-1\}, \quad \forall \tau \geq T : \\ x_T \in \mathcal{G}, \quad x_t \notin \mathcal{U}, \quad x_\tau \in \mathcal{F}$$



a: Stability b: ROA c: Safety d: SWA e: Reachability f: RWA g: RSWA h: RAR

Synthesis of control certificates for complex tasks

- dynamical models with inputs (a.k.a., external non-determinism)

$$x^+ = f(x, u)$$

→ synthesis of "control certificates"

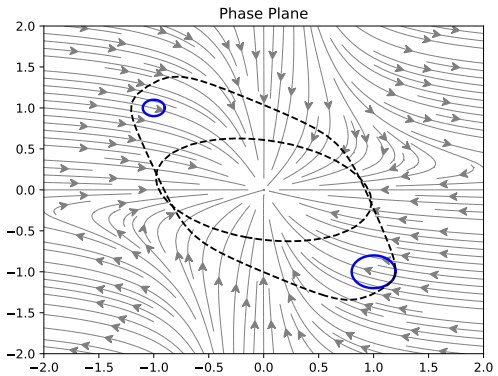
	N_s	N_u	Property	Neurons	Activations	T (s)			Success (%)
						min	μ	max	S
1	2	0	Stability	[6]	$[\varphi_2]$	0.01 ($\approx$ 0.00)	0.16 (0.15)	1.50 (1.48)	100
2	3	0	Stability	[8]	$[\varphi_2]$	0.28 ($\approx$ 0.00)	2.22 (0.45)	12.57 (3.31)	100
3	2	2	Stability	[4]	$[\varphi_2]$	0.07 (0.01)	0.19 (0.02)	0.47 (0.04)	100
4	2	2	Stability	[5]	$[\varphi_2]$	0.09 (0.01)	0.26 (0.02)	0.54 (0.03)	100
5	2	0	ROA	[5]	$[\sigma_{\text{soft}}]$	0.21 (0.12)	14.09 (12.59)	25.32 (22.13)	40
6	3	3	ROA	[8]	$[\varphi_2]$	1.24 (0.02)	39.08 (0.03)	287.89 (0.04)	100
7	2	0	Safety	[15]	$[\sigma_{\text{t}}]$	0.44 (0.35)	3.36 (2.90)	7.61 (7.11)	100
9	8	0	Safety	[10]	$[\varphi_1]$	12.63 (7.71)	51.97 (32.75)	70.59 (44.66)	70
10	3	1	Safety	[15]	$[\sigma_{\text{t}}]$	1.57 (0.19)	11.87 (2.50)	51.08 (7.52)	90
11	3	0	SWA	[6], [5]	$[\varphi_2]$, $[\sigma_{\text{t}}]$	0.19 (0.05)	2.46 (0.100)	12.10 (0.20)	90
12	2	0	SWA	[5], [5, 5]	$[\varphi_2]$, $[\sigma_{\text{sig}},\varphi_2]$	0.13 (0.06)	0.27 (0.14)	0.39 (0.20)	100
13	2	1	SWA	[8], [5]	$[\varphi_2]$, $[\varphi_2]$	0.06 (0.03)	0.20 (0.10)	0.58 (0.24)	90
14	3	1	SWA	[10], [8]	$[\varphi_2]$, $[\sigma_{\text{t}}]$	4.06 (0.87)	19.81 (2.73)	103.49 (7.23)	90
15	2	0	RWA	[4]	$[\varphi_2]$	0.14 (0.09)	1.81 (1.75)	4.70 (4.63)	100
16	3	0	RWA	[16]	$[\varphi_2]$	1.36 (0.09)	14.10 (0.14)	72.97 (0.20)	90
17	2	1	RWA	[4, 4]	$[\sigma_{\text{sig}},\varphi_2]$	0.59 (0.27)	6.82 (3.32)	20.07 (11.46)	100
18	3	1	RWA	[5]	$[\varphi_2]$	0.46 (0.11)	16.06 (5.81)	72.47 (44.64)	80
19	2	2	RWA	[5]	$[\sigma_{\text{sig}}]$	0.69 (0.40)	1.38 (0.94)	2.14 (1.90)	100
20	2	0	RSWA	[4]	$[\varphi_2]$	0.19 (0.03)	1.29 (1.04)	3.79 (3.37)	100
21	3	0	RSWA	[16]	$[\varphi_2]$	4.81 (0.13)	27.14 (0.19)	80.95 (0.25)	100
22	2	0	RSWA	[5, 5]	$[\sigma_{\text{sig}},\varphi_2]$	1.52 (0.06)	4.45 (0.19)	10.97 (0.35)	100
23	2	1	RSWA	[8]	$[\varphi_2]$	0.21 (0.05)	0.67 (0.25)	1.19 (0.91)	100
24	2	2	RSWA	[5, 5]	$[\sigma_{\text{sig}},\varphi_2]$	0.98 (0.16)	1.23 (0.28)	1.61 (0.46)	100
25	2	0	RAR	[6], [6]	$[\sigma_{\text{soft}}]$, $[\varphi_2]$	6.65 (1.08)	24.74 (6.46)	77.80 (15.06)	100
26	2	2	RAR	[6, 6], [6, 6]	$[\sigma_{\text{sig}},\varphi_2]$, $[\sigma_{\text{sig}},\varphi_2]$	5.13 (1.34)	26.99 (9.90)	101.23 (60.14)	100

Synthesis of control certificates for complex tasks

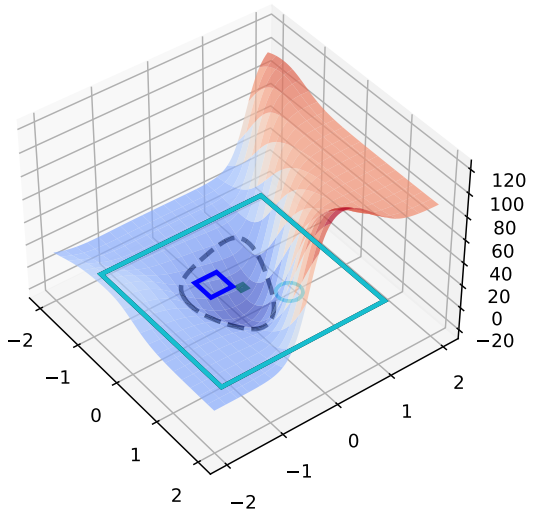
- dynamical models with inputs (a.k.a., external non-determinism)

$$x^+ = f(x, u)$$

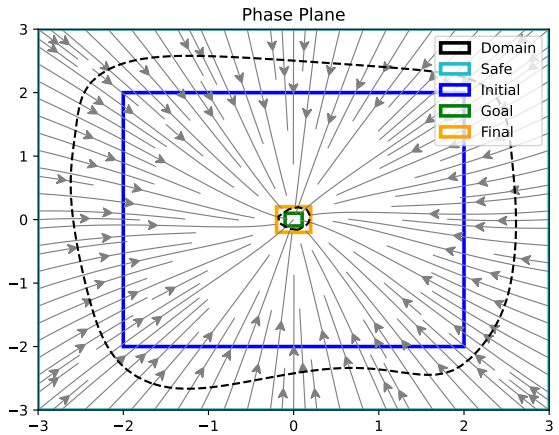
$\rightarrow$ synthesis of "control certificates"



ROA for NL model,
non-poly Lyapunov,
2 disjoint initial sets



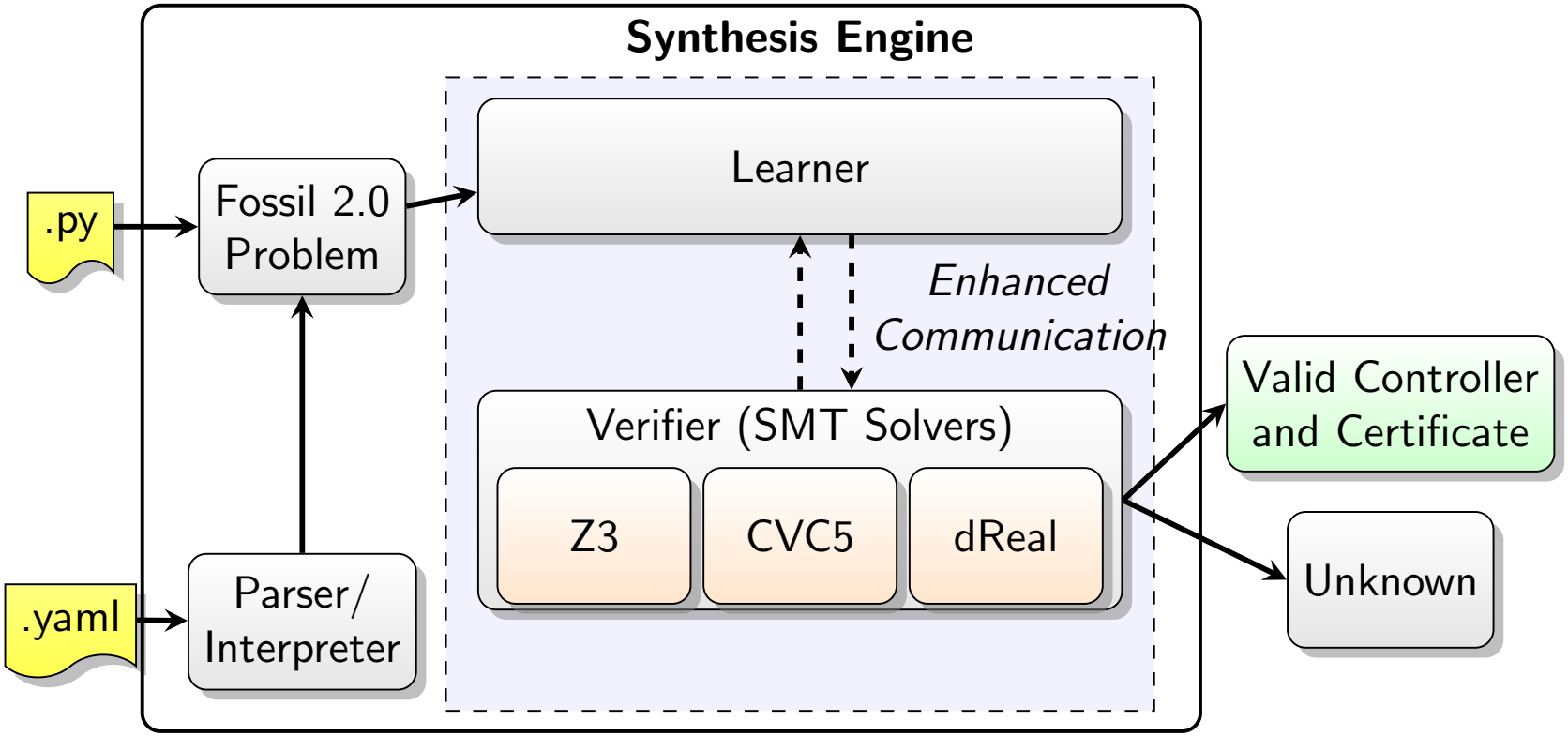
RWA: reach-while-avoid



RAR certificate for closed-loop NL model

dashed lines: level sets; dark blue: $\mathcal{I}$; light blue: $\mathcal{S}$; green: $\mathcal{G}$; orange: $\mathcal{F}$

Software for Neural Synthesis - Fossil 2.0



`github.com/oxford-oxcav/fossil`

Ranking certificates as supermartingales

- almost-sure (w.p. 1) countable-state program termination
- can be extended to
  - continuous state spaces
  - control synthesis
  - quantitative termination (e.g., via supermartingale inequalities)

Ranking certificates as supermartingales

- almost-sure (w.p. 1) countable-state program termination
- can be extended to
  - continuous state spaces
  - control synthesis
  - quantitative termination (e.g., via supermartingale inequalities)

. . . and to whole-LTL quantitative model checking and design of gMDP

Proof rule for almost-sure Streett acceptance

- consider Markov chain $\mathtt{x}^+ = f(\mathtt{x}, \mathtt{w})$, along with Streett pair:

$$\mathsf{GF}(A) \implies \mathsf{GF}(B) \qquad\qquad A, B \subseteq \mathsf{State}$$

Streett Supermartingale Proof Rule [AA et al, CAV24]

$$\exists \text{ function } V : \mathsf{State} \to \mathbb{R}_{\geq 0},$$

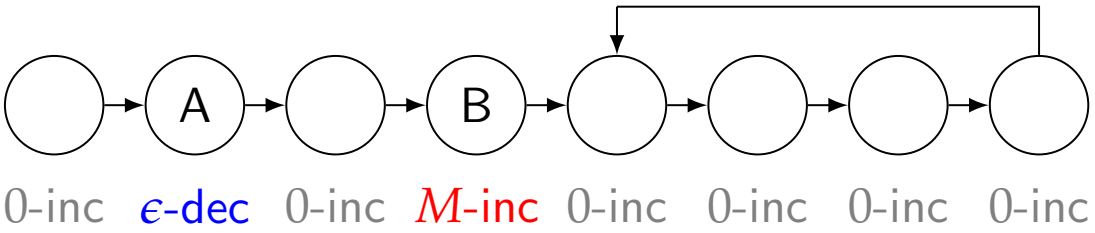
- **In B:** allow V to increase in expectation by M (positive const)
- **In $A \setminus B$:** decrease V in expectation by ϵ (positive const)
- **Otherwise:** ensure V does not increase in expectation

V is **non-negative** and **almost** a supermartingale

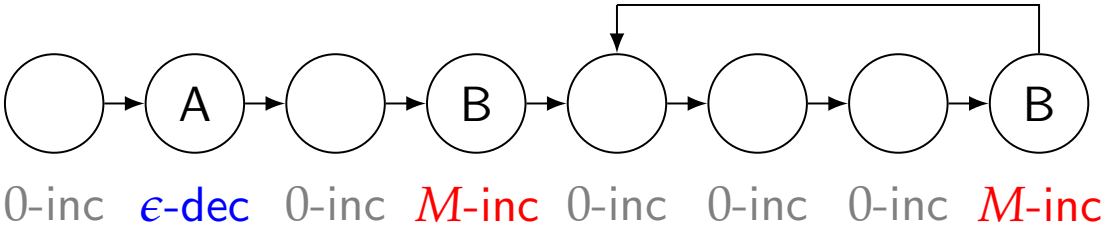
allow increase in B

Proof rule for almost-sure Streett acceptance

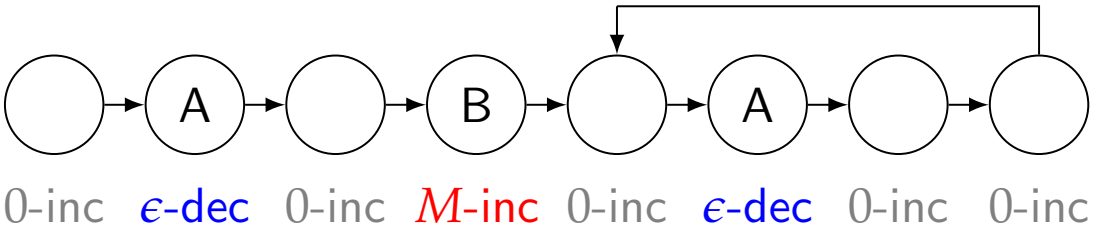
$$\mathrm{GF}(\mathrm{A}) \implies \mathrm{GF}(\mathrm{B})?$$



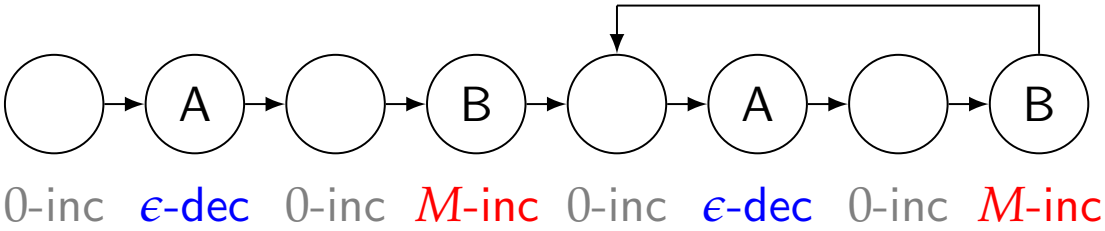
✓ finite A's, finite B's



✓ finite A's, infinite B's



✗ infinite A's, finite B's



✓ infinite A's, infinite B's

Experiments

- I - invariance generation
- V - verification (certificate synthesis)
- C - control synthesis

Benchmark	ω-Regular Specification	Output	Time [s]
`EvenOrNegative`	$\mathsf{GF}(x \text{ even}) \vee \mathsf{FG}(x < 0)$	V	0.09
`SafeRWalk1`	$\mathsf{G}(x < 100)$	VIC	1.09
`PersistRW`	$\mathsf{FG}(x \leq 10)$	VI	1.16
`RecurRW`	$\mathsf{GF}(x > 100)$	VI	1.49
`SafeRWalk2`	$\mathsf{G}(x \geq 10)$	VIC	1.09
`GuaranteeRW`	$\mathsf{G}(x \geq -10) \rightarrow \mathsf{F}(x \geq 10^3)$	VI	5.61
`Temperature1`	$\mathsf{FG}(\neg \mathsf{Hot} \wedge \neg \mathsf{Cold})$	VIC	4.11
`Temperature2`	$\mathsf{GF}(x \leq 30) \wedge \mathsf{G}(x \leq 60)$	VI	28.93
`Temperature3`	$\mathsf{G}(\mathsf{Safe}) \wedge [\mathsf{GF}(\mathsf{Cold}) \rightarrow \mathsf{GF}(\mathsf{Hot})]$	VIC	28.58
`Temperature4`	$\mathsf{G}(\mathsf{Safe}) \wedge [\mathsf{GF}(\mathsf{Cold}) \rightarrow \mathsf{GF}(\mathsf{Hot})]$	VC	4.64
`FinMemoryControl`	$\mathsf{GF}(x \leq 0) \rightarrow \mathsf{GF}(x \geq 100)$	VIC	16.73

Outline

1. Context
2. Neural Proofs - Formal Synthesis of Neural Certificates
3. Synthesis of Formal Abstractions

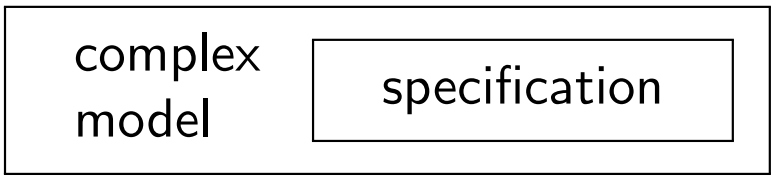
Outline

1 Context

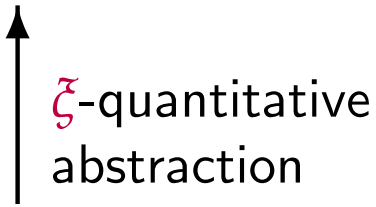
2 Neural Proofs - Formal Synthesis of Neural Certificates

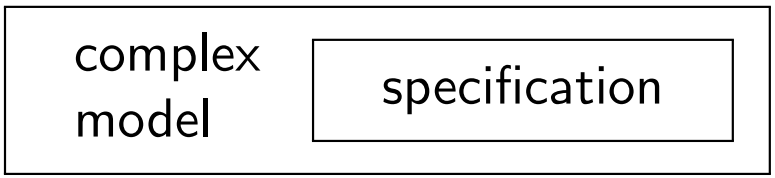
3 Synthesis of Formal Abstractions

Formal abstractions

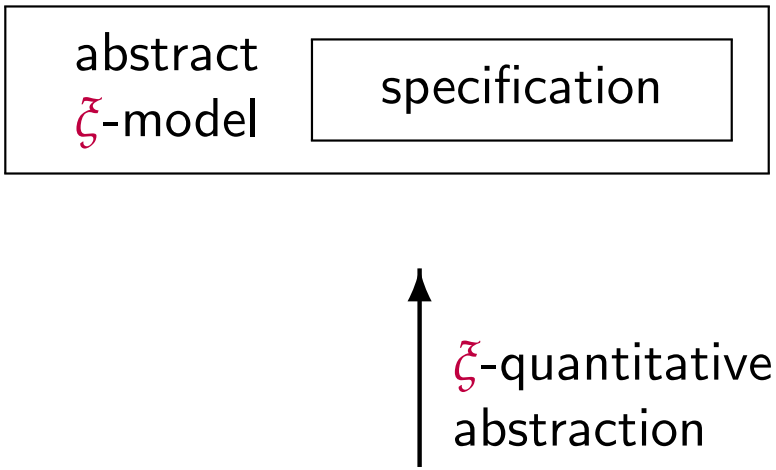


Formal abstractions

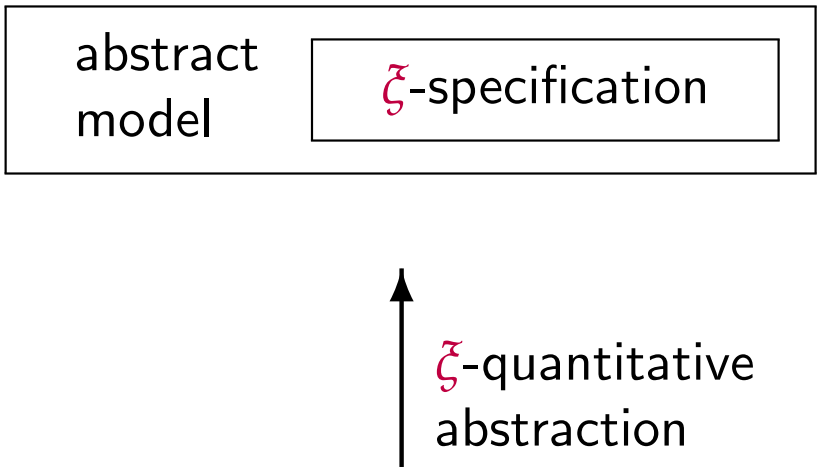




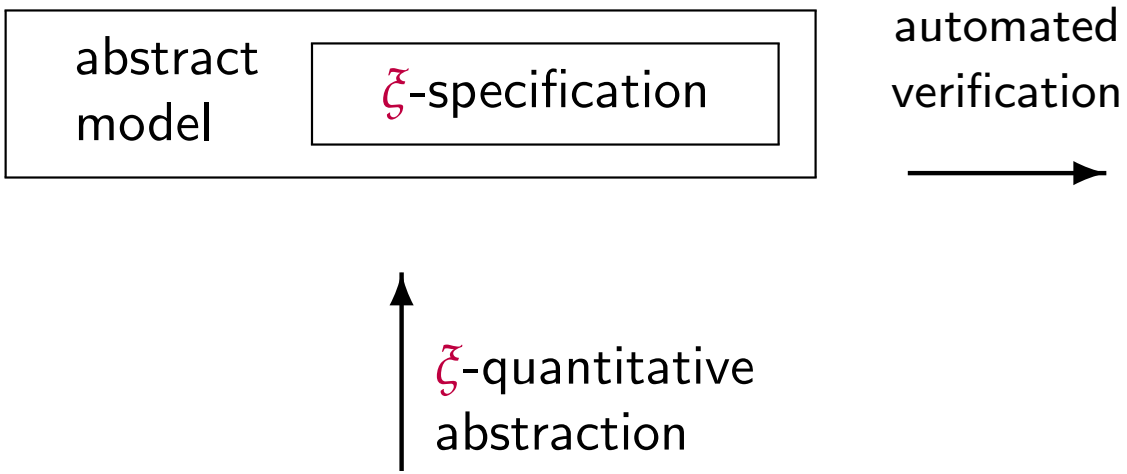
Formal abstractions



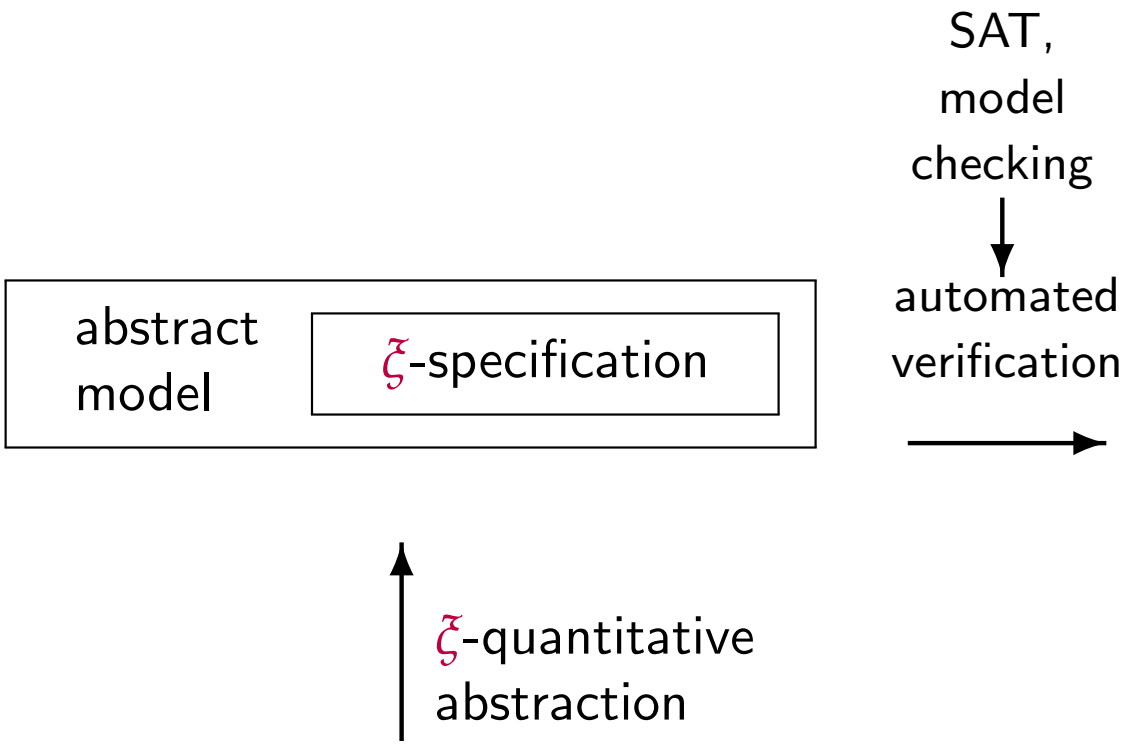
Formal abstractions



Formal abstractions



Formal abstractions



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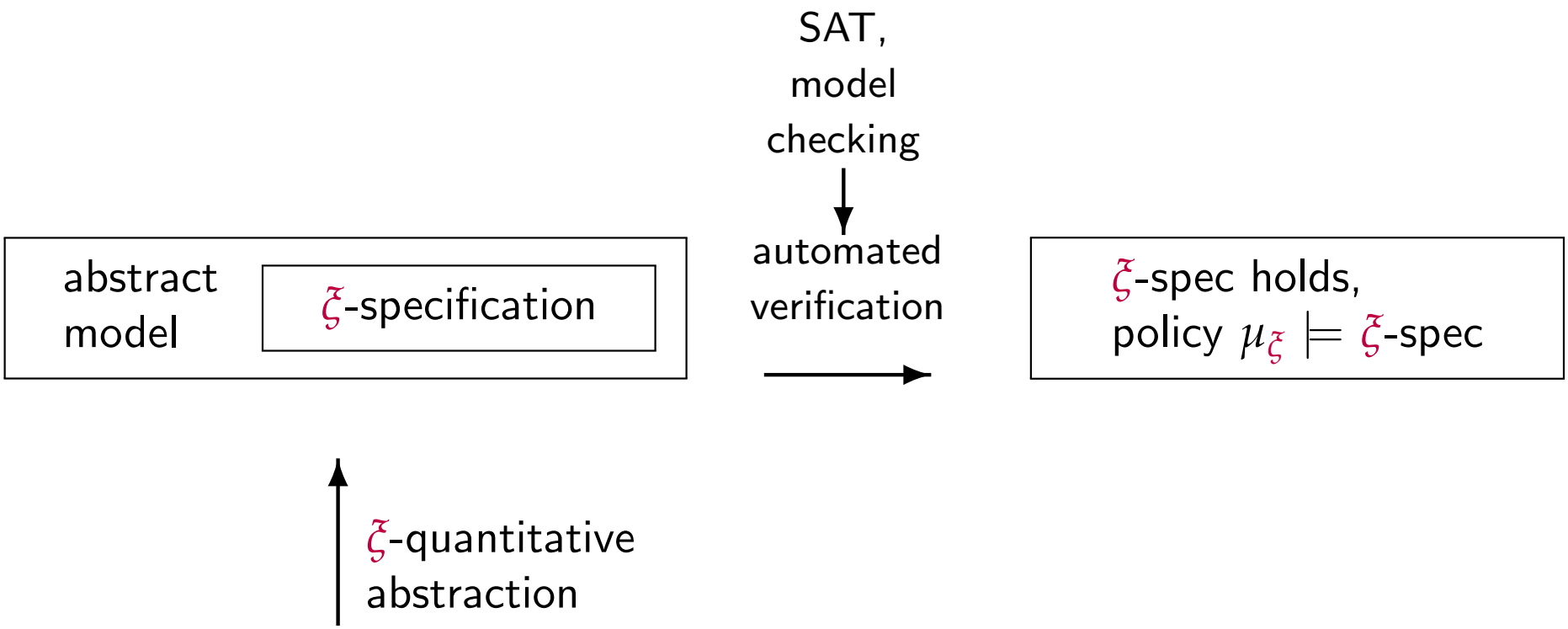

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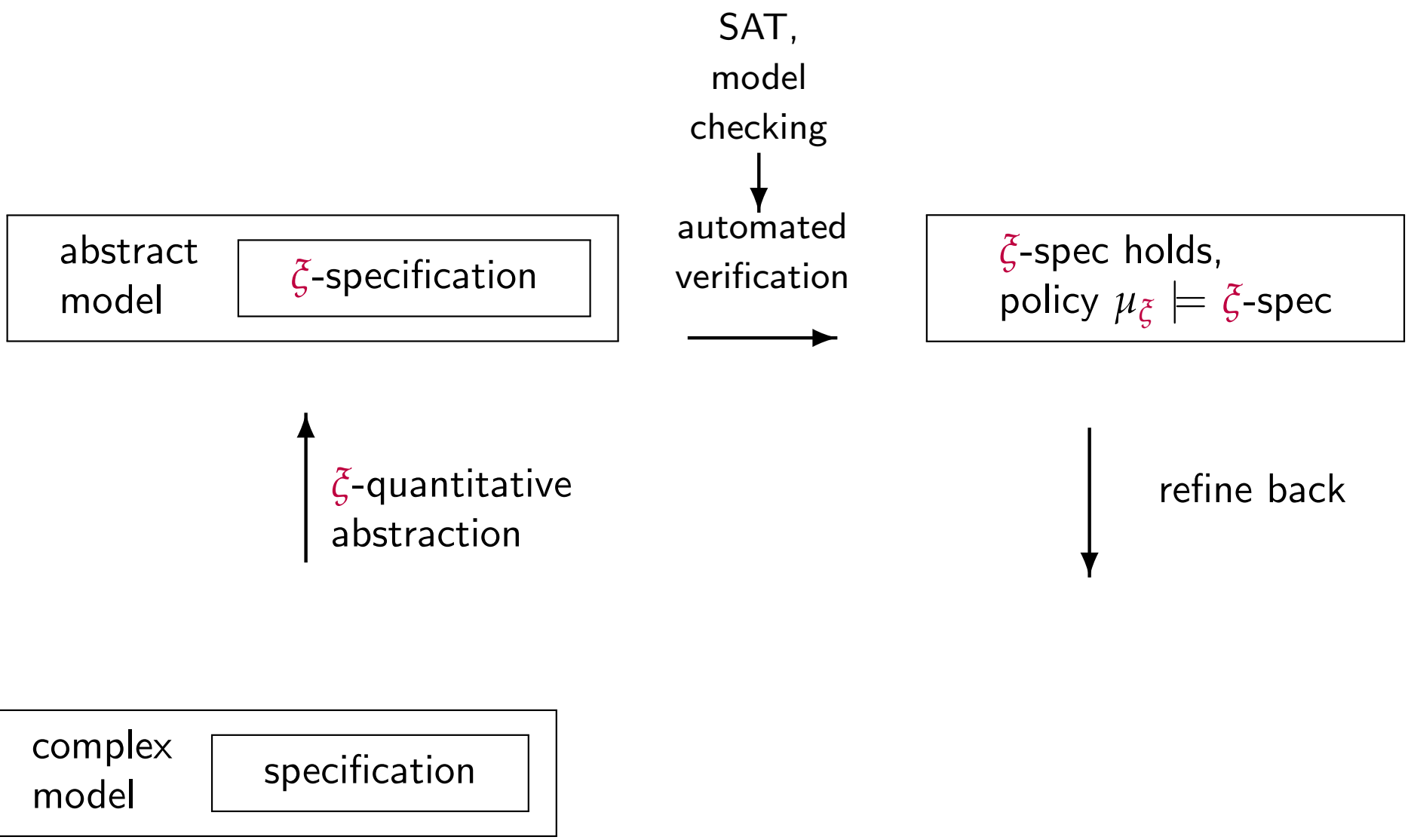

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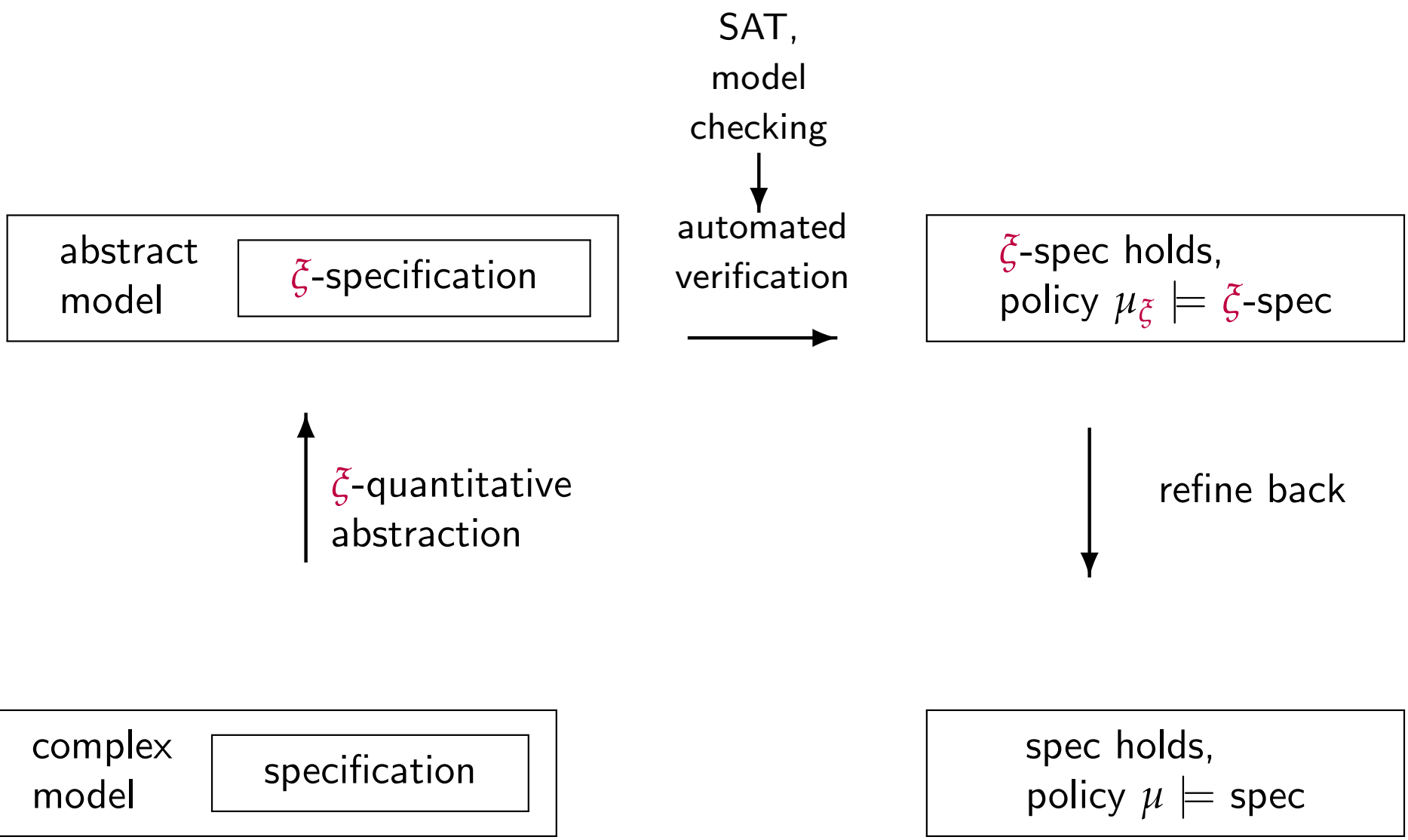
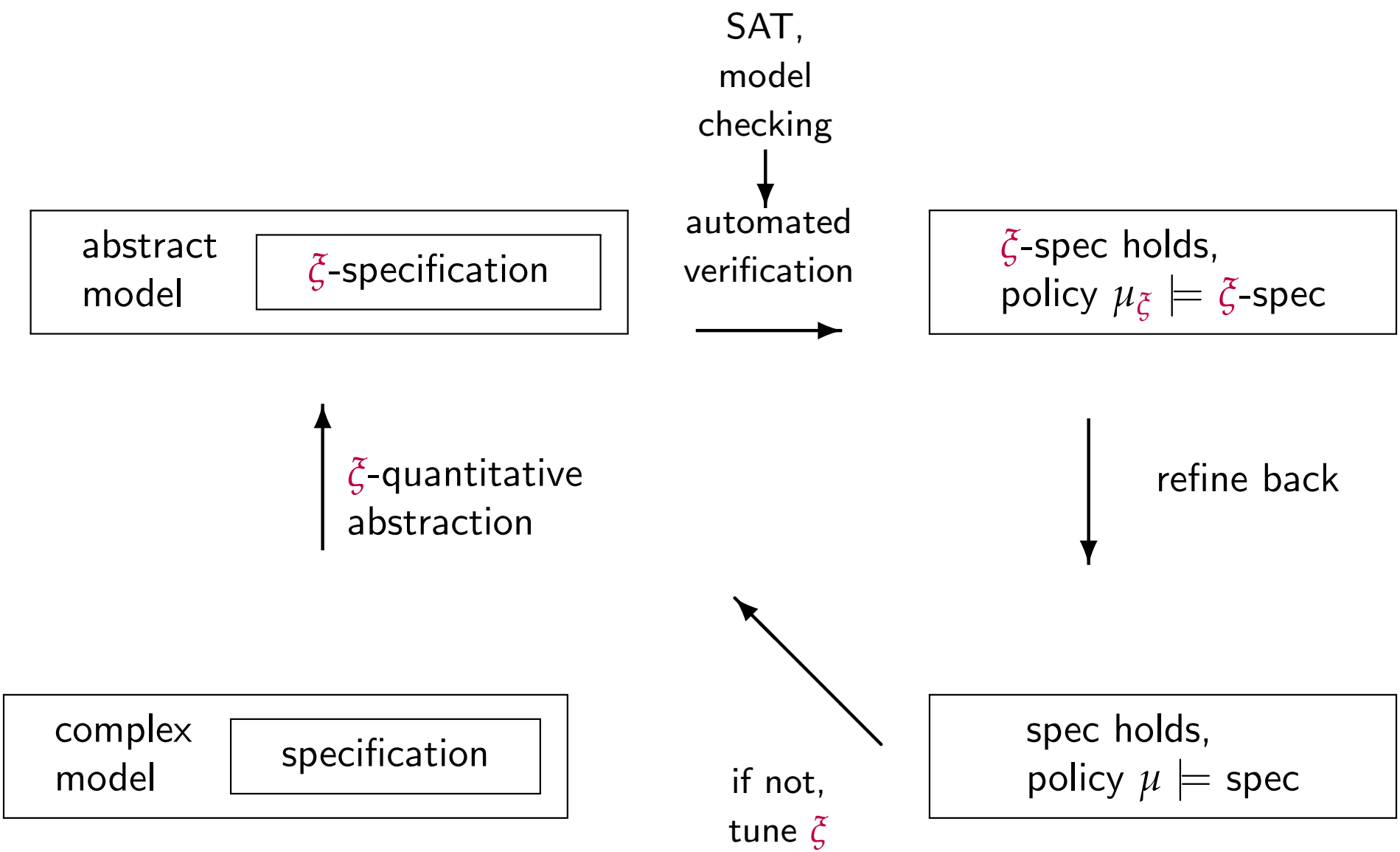


Formal abstractions



1. Context

2. Neural Proofs - Formal Synthesis of Neural Certificates

3. Synthesis of Formal Abstractions

Thank you for your attention

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Almost-sure proof rules

so far we have seen that:

- ranking supermartingales for almost-sure reachability **(Chakarov/Sankaranarayanan, CAV'13)**
- almost sure persistence & recurrence **(Chakarov/Voronin/Sankaranarayanan, TACAS'16)**
- Streett supermartingales for almost-sure reactivity conditions **(AA et. al., CAV'24)**

and indeed **many other similar proof rules** ...

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- now, we can generalise any **almost-sure proof rule** to **quantitative** supermartingale certificates:

$$\boxed{\mathbb{P}\left(\text{ Specification }\right) \geq p}$$

- this is done via a "decomposition"

  (AA et. al., CAV'25)

Decomposing Quantitative Probabilistic Verification

- soundness of decomposition - suppose:
  1. $\mathbb{P}(\text{ Specification } \mid \mathbf{G}\ I\ ) = 1$
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- special case:
  1. if $\mathbb{P}(\text{ Streett acceptance } |\ \mathbf{G}\ I\ ) = 1$
     (e.g., via Streett supermartingale & supporting invariant, [AA et. al., CAV'24])
  2. if $\mathbb{P}(\ \mathbf{G}\ I\ ) \geq p$
     (e.g., via stochastic invariant (supermartingale) [Kushner, 1965] or repulsing supermartingales [Chatterjee et.al., POPL'17][Takisaka et.a., ATVA'18])
- then $\mathbb{P}(\text{ Streett acceptance }) \geq p$

Completeness of Decomposition

- **question:** does there always exist a region $I \subseteq$ State such that:
  1. $\mathbb{P}(\text{ Specification } \mid \mathbf{G}\ I\ ) = 1$,
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- **poositive answer:** suppose L is a shift-invariant specification[5], then the following holds:

Theorem (ε-completeness for gMC)

For every $\varepsilon > 0$, there exists a region I such that

1. $\mathbb{P}(\ L \mid \mathbf{G}\ I\ ) = 1$
2. $\mathbb{P}(\ \mathbf{G}\ I\ ) \geq \mathbb{P}(\ L\ ) - \varepsilon$

Theorem (Completeness for finite MC)

There exists a region I such that

1. $\mathbb{P}(\ L \mid \mathbf{G}\ I\ ) = 1$
2. $\mathbb{P}(\ \mathbf{G}\ I\ ) = \mathbb{P}(\ L\ )$

[5] A language is invariant under addition/deletion of any finite prefix to an infinite trajectory